Degeneracy of the b-boundary in General Relativity

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Abstract

The b-boundary construction by B. Schmidt is a general way of providing a boundary to a manifold with connection [12]. It has been shown to have undesirable topological properties however. C. J. S. Clarke gave a result showing that for space-times, non-Hausdorffness is to be expected in general [3], but the argument contains some errors. We show that under somewhat different conditions on the curvature, the b-boundary will be non-Hausdorff, and illustrate the degeneracy by applying the conditions to some well known exact solutions of general relativity.

1 Introduction

A serious limitation in our understanding of singularities in general relativity is the fact that singularities by definition are not parts of the space-time manifold. So in order to study the structure of singularities we would like to have some procedure for attaching an abstract boundary set containing the singular points to a space-time. At the very least the extended space-time should have a suitable topology making it possible to make statements like 'close to the singularity' mathematically precise.

One of the candidates is the b-boundary construction by B. Schmidt which works for any manifold with connection [12], and in the Lorentzian case it can be shown to be well-defined and locally complete [13]. However, for the FLRW and Schwarzschild space-times, the boundary is not Hausdorff separated from interior points [1, 7]. This is a serious drawback since all points in space-time are then 'close' to a given boundary point, making all statements about neighbourhoods of the singularity useless.

The b-boundary structure is closely related to the singular holonomy group [2]. The methods used by Bosshard [1] and Johnson [7] are heavily dependent on the specific geometry of the FLRW and Schwarzschild space-times, based on the study of the boundary of two-dimensional sections. Clarke used a more general

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approach to find sufficient conditions for the topology to be non-Hausdorff [3]. The condition involves the asymptotic behaviour of the Riemann tensor and its inverse and derivative in a parallel propagated frame along a curve ending at the boundary point.

The argument in [3] contains some errors however. We show that under somewhat different conditions on the Riemann tensor and its inverse and derivative, the boundary fibres of the frame bundle are degenerate. We also confirm that the conditions hold in the FLRW (with expansion factor t^c with $c \in (0, 1)$, which is a bit more general than in [3]), Kasner, Schwarzschild, Reissner-Nordström and Tolman-Bondi space-times.

Our reasoning will depend a lot on the work by Clarke [3], the most essential difference being that we choose to work with small circles instead of squares in §3 and that we use a stronger restriction initially on the derivative of the Riemann tensor. The outline of the paper is as follows. In §2, we introduce some notation and definitions. In §3 we approximate Lorentz transformations resulting from parallel propagation along small circles in terms of the Riemann tensor, and in §4 we use these results to find a curve generating a given Lorentz transformation by parallel propagation. §5 is concerned with singular holonomy and gives the connection to the b-boundary, and we illustrate the implications for some well known space-times in §6. We also discuss some other contributions to the singular holonomy group in §7.

2 Preliminaries

Throughout this paper, (M, g) is a space-time, i.e. a smooth 4-dimensional connected orientable and Hausdorff manifold M with a smooth metric g of signature $(-+++)$. The construction of the b-boundary may be carried out in different bundles over M (see Refs. [12], [4] or [6] for some background). Here we choose to work in the bundle of pseudo-orthonormal frames OM, consisting of all pseudo-orthonormal frames at all points of M. OM is a principal fibre bundle with the Lorentz group $\mathcal{L}$ as its structure group. We write the right action of an element $\mathbf{L} \in \mathcal{L}$ as $R_{\mathbf{L}} : E \mapsto E\mathbf{L}$ for $E \in OM$. We will, a bit sloppily, restrict attention to one of the connected components of OM and the component of identity in $\mathcal{L}$ using the same notation.

From the fibre bundle structure of OM, we have a canonical 1-form θ which is $\mathbb{R}^4$-valued, and from the metric on M we construct the connection form ω, which takes values in the Lie algebra $\mathfrak{l}$ of $\mathcal{L}$ [8]. The Schmidt metric on OM is the Riemannian metric G given by

$$G(X, Y) := \langle \theta(X), \theta(Y) \rangle_{\mathbb{R}^4} + \langle \omega(X), \omega(Y) \rangle_{\mathfrak{l}} \tag{1}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{l}}$ are Euclidian inner products with respect to fixed bases in $\mathbb{R}^4$ and $\mathfrak{l}$, respectively [12, 4].

If κ is a curve in the bundle of pseudo-orthonormal frames OM, we denote the b-length of κ by $l(\kappa)$. By a slight abuse of notation we will also write $l(\gamma, E_0)$

for the b-length, or generalised affine parameter length, of a curve $\gamma(t)$ in M, with respect to a given frame E_0 at some point on γ. The definition is

$$l(\gamma, E_0) := \int \left(\sum_{i=1}^{n} (\mathbf{V}^i)^2 \right)^{\frac{1}{2}} \mathrm{d}t, \tag{2}$$

where $\mathbf{V}^i$ are the components of the tangent vector of γ with respect to the frame E obtained by parallel propagation of E_0 along γ. The notation is motivated by the fact that $l(\gamma, E_0)$ is the same as the b-length of the horizontal lift of γ through E_0 in OM. We also write $d(E, F)$ for the b-metric distance between two points E and F in OM, and $B_r(E)$ for the open ball in OM with centre at E and radius r.

The Schmidt metric was used by Schmidt [12] to construct a boundary, the b-boundary, of the base manifold M, providing endpoints for all b-incomplete inextendible curves. Basically the procedure is as follows.

1. Construct the Cauchy completion $\overline{OM}$ of OM and extend the group action to $\overline{OM}$.
2. Let $\overline{M}$ be the set of orbits of $\mathcal{L}$ in $\overline{OM}$, and define a projection $\overline{\pi} : \overline{OM} \to \overline{M}$ taking a point in $\overline{OM}$ to the orbit through the point.
3. $\overline{M}$ is then a topological space with the topology inherited from $\overline{OM}$ via $\overline{\pi}$, and we may identify $\overline{\pi}(OM)$ with M.
4. Define the b-boundary as $\partial M = \overline{M} \setminus M$.

The topological space $\overline{OM}$ is no longer a fibre bundle since the action of $\mathcal{L}$ might be non-free on a boundary 'fibre' (orbit). We quantify the boundary fibre degeneracy by defining the singular holonomy group as

$$\Phi^s_{OM}(E) := \{\mathbf{L} \in \mathcal{L}; E\mathbf{L} = E\}, \tag{3}$$

for $E \in \overline{\pi}^{-1}(p)$ with $p \in \partial M$ [2]. It follows that the boundary fibre $\overline{\pi}^{-1}(p)$ is homeomorphic to $\mathcal{L}/\Phi^s_{OM}(E)$. We say that the boundary fibre is degenerate if the singular holonomy group is nontrivial, and totally degenerate if the singular holonomy group is the whole Lorentz group $\mathcal{L}$. The importance of total degeneracy is illustrated by the following result from [3].

Proposition 1. *If $p \in \partial M$ with $\overline{\pi}^{-1}(p)$ totally degenerate, then every neighbourhood of p in $\overline{M}$ contains all null geodesics in M ending at p. In particular, $\overline{M}$ is not Hausdorff.*

In what follows we will need various norms, given a fixed frame $E \in OM$. We use bold symbols for the array of frame components of a tensor in the frame E. For tangent vectors X, we define the norm $|\mathbf{X}|$ to be the Euclidian norm of the frame component array $\mathbf{X}$, and similarly for cotangent vectors. In the Lie group and Lie algebra, we use the Euclidian norm with respect to a fixed basis, and for general tensors T we use the mapping norm, e.g.

$$\|\mathbf{T}\| := \sup_{|\mathbf{X}|=|\mathbf{Y}|=1} |T_{ij}X^iY^j| \tag{4}$$

for a covariant 2-tensor T.

3 Parallel propagation and the Riemann tensor

In this section we calculate a first approximation to the Lorentz transformation generated by parallel propagation around a small circle. First we construct a disc with suitable properties. Let $f : D_l \to M$, where

$$D_l := \{(x, y) \in \mathbb{R}^2;\, x^2 + y^2 \leq l^2\}, \tag{5}$$

and put

$$X(x, y) := f_* \frac{\partial}{\partial x}\Big|_{f(x,y)} \tag{6}$$

$$Y(x, y) := f_* \frac{\partial}{\partial y}\Big|_{f(x,y)}. \tag{7}$$

Let (r, θ) be polar coordinates on D_l, i.e. $x = r\cos\theta$ and $y = r\sin\theta$, and put

$$V(x, y) := f_* \frac{\partial}{\partial r}\Big|_{f(x,y)} \tag{8}$$

$$Z(x, y) := f_* \frac{\partial}{\partial \theta}\Big|_{f(x,y)}. \tag{9}$$

Then

$$V = \cos\theta\, X + \sin\theta\, Y \tag{10}$$

$$Z = -r\sin\theta\, X + r\cos\theta\, Y. \tag{11}$$

Pick a pseudo-orthonormal frame $E(0, 0)$ at $p := f(0, 0)$, and define $E(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, by parallel propagating $E(0, 0)$ along the radial curves $\rho_\theta : s \mapsto f(s\cos\theta, s\sin\theta)$ for each $\theta \in [0, 2\pi)$. Similarly, let $F(x, y)$ be defined by parallel propagating $E(r, 0)$ along the circular curve $o_r : s \mapsto f(r\cos s, r\sin s)$ for each $r \in [0, l]$. Let $\mathbf{L}(x, y)$ be the Lorentz transformation taking $E(x, y)$ to $F(x, y)$, i.e. $F = E\mathbf{L}$. From now on, bold symbols denote component arrays with respect to the frame E.

Lemma 1. *The Lorentz transformation* $\mathbf{L}$ *is given by*

$$\mathbf{L} = \exp -\int_0^\theta \int_0^r \mathbf{R}(\mathbf{V}, \mathbf{Z})\, \mathrm{d}r\, \mathrm{d}\theta \tag{12}$$

where exp *is the exponential map* $\mathfrak{l} \to \mathcal{L}$.

Proof. Since F is parallel along each o_r,

$$\nabla_Z F = (\nabla_Z E)\mathbf{L} + E\nabla_Z \mathbf{L} = 0. \tag{13}$$

We may view $\mathbf{L}$ on each o_r as a curve in the Lorentz group $\mathcal{L}$ parameterised by θ. Then

$$E\dot{\mathbf{L}}\mathbf{L}^{-1} = -\nabla_Z E \tag{14}$$

where the dot denotes differentiation with respect to θ. Now let λ be the curve in the Lie algebra corresponding to $\mathbf{L}$ by right translation, i.e. λ corresponds to the right-invariant vector field equal to $\dot{\mathbf{L}}$ at $\mathbf{L}$ by $\dot{\mathbf{L}} = \lambda\mathbf{L}$. (It might seem more natural to choose left translation, but then we would have to solve for $\mathbf{L}^{-1}$ instead.) Thus

$$E\lambda = -\nabla_Z E. \tag{15}$$

Differentiating with respect to V and using that $\nabla_V E = 0$ and

$$\nabla_V \nabla_Z E = R(V, Z)E \tag{16}$$

we get

$$\frac{\partial\lambda}{\partial r} = -\mathbf{R}(\mathbf{V}, \mathbf{Z}) \tag{17}$$

in the frame E. Integrating and solving $\dot{\mathbf{L}} = \lambda\mathbf{L}$ gives

$$\mathbf{L} = \exp -\int_0^\theta \int_0^r \mathbf{R}(\mathbf{V}, \mathbf{Z}) \, \mathrm{d}r \, \mathrm{d}\theta. \tag{18}$$

□

Corollary 1. *The Lorentz transformation Λ generated by parallel propagation counterclockwise around the boundary of $f(D_l)$ is given by*

$$\Lambda = \exp -\int_0^{2\pi} \int_0^l \mathbf{R}(\mathbf{V}, \mathbf{Z}) \, \mathrm{d}r \, \mathrm{d}\theta = \exp -\iint_{D_l} \mathbf{R}(\mathbf{X}, \mathbf{Y}) \, \mathrm{d}\sigma \tag{19}$$

where $\mathrm{d}\sigma$ is the area element of D_l with respect to the metric $\mathrm{d}x^2 + \mathrm{d}y^2$.

Proof. The first expression follows immediately by letting $r \to l$ and $\theta \to 2\pi$ in Lemma 1. Using (10) and (11) and the symmetries of the Riemann tensor we get

$$R(V, Z) = rR(X, Y) \tag{20}$$

and hence the second formula. □

Let γ be the loop at p obtained by following the radial curve ρ_0, the boundary o_l of the disc in the counterclockwise direction, and back again along ρ_0^{-1}. Then parallel propagation along γ generates Λ since E is parallel along ρ_0. Suppose that f is chosen such that the radial curves ρ_θ are geodesics and $|\mathbf{X}| = |\mathbf{Y}| = 1$ at p, i.e. f is basically the exponential map $T_pM \to M$, restricted to

$$\{(xX_p + yY_p \in T_pM; x^2 + y^2 \leq l^2\}.$$

We can then approximate Λ by an expression involving the value of $\mathbf{R}$ only at p. The essential thing here is the length estimate.

Note. Whilst f is smooth by construction, it need not be an embedding or even 1–1. In such a case, E is not a frame field on D_l, but the construction still works.

Lemma 2. *Suppose that $\nabla_V V = 0$, $|\mathbf{X}| = |\mathbf{Y}| = 1$ at p, and $l > 0$ is sufficiently small for there to be an $\alpha < 1$ such that*

$$l^2 \|\mathbf{R}\|_{D_l} \leq 10^{-3}\alpha \tag{I}$$

$$l^3 \|\nabla\mathbf{R}\|_{D_l} \leq 10^{-6}\alpha^2 \tag{II}$$

where $\|\cdot\|_{D_l} := \sup_{f(D_l)}\|\cdot\|$. Then

$$\left\|\Lambda - \delta + \pi l^2\, \mathbf{R}(\mathbf{X}, \mathbf{Y})|_p\right\| < 10^{-5}\alpha^2 \tag{21}$$

where δ is the identity element of $\mathcal{L}$, and the b-length of γ is less than $9l$.

Proof. Note that $\nabla_V V = 0$ implies that $|\mathbf{V}| = 1$ on the whole disc. First we need estimates for $|\mathbf{Z}|$ and $|\nabla_{\mathbf{V}}\mathbf{Z}|$. Since $[V, Z] = 0$,

$$\nabla_V^2 Z = \nabla_V \nabla_Z V = R(V, Z)V, \tag{22}$$

so

$$\nabla_{\mathbf{V}}\mathbf{Z} = \nabla_{\mathbf{V}}\mathbf{Z}|_p + \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}|_{r=\xi_1}\, r \tag{23}$$

and

$$\mathbf{Z} = \mathbf{Z}_p + \nabla_{\mathbf{V}}\mathbf{Z}|_p\, r + \frac{1}{2}\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}|_{r=\xi_2}\, r^2 \tag{24}$$

for some $\xi_1, \xi_2 \in [0, r]$. But from (11), $\mathbf{Z}_p = 0$ and

$$\nabla_{\mathbf{V}}\mathbf{Z}|_p = -\sin\theta\, \mathbf{X}_p + \cos\theta\, \mathbf{Y}_p, \tag{25}$$

so

$$|\mathbf{Z}| \leq r + \frac{1}{2}\|\mathbf{R}\|_{D_l}\, |\mathbf{Z}|_{r=\xi_2}\, r^2 \leq r + \frac{\alpha}{2000}|\mathbf{Z}|_{r=\xi_2}, \tag{26}$$

and since $\alpha < 1$,

$$|\mathbf{Z}| < \frac{2000}{1999}\, r, \tag{27}$$

and

$$|\nabla_{\mathbf{V}}\mathbf{Z}| < 1 + \frac{2}{1999}\,\alpha < \frac{2001}{1999}. \tag{28}$$

Put

$$\lambda := -\int_0^{2\pi}\!\!\int_0^l \mathbf{R}(\mathbf{V},\mathbf{Z})\,\mathrm{d}r\,\mathrm{d}\theta. \tag{29}$$

Then

$$\|\lambda\| \le 2\pi\|\mathbf{R}\|_{D_l}\int_0^l |\mathbf{Z}|\,\mathrm{d}r < \frac{\alpha}{300}, \tag{30}$$

so

$$\|\Lambda - \delta - \lambda\| \le \sum_{k=2}^{\infty}\frac{\|\lambda\|^k}{k!} < \|\lambda\|^2\sum_{k=0}^{\infty}\frac{\|\lambda\|^k}{2^k} < \frac{\alpha^2}{80000}. \tag{31}$$

Next we replace the integral in λ with an expression involving only the value of the Riemann tensor at the origin. The mean value theorem gives

$$\mathbf{R}(\mathbf{V},\mathbf{Z}) = \mathbf{R}(\mathbf{V},\mathbf{Z})|_p + \nabla_{\mathbf{V}}\big(\mathbf{R}(\mathbf{V},\mathbf{Z})\big)|_{r=\xi_3}\, r \tag{32}$$

for some $\xi_3 \in [0,r]$. Since $\mathbf{Z}_p = 0$ and $\nabla_V V = 0$,

$$\mathbf{R}(\mathbf{V},\mathbf{Z}) = (\nabla_{\mathbf{V}}\mathbf{R})(\mathbf{V},\mathbf{Z})|_{r=\xi_3}\, r + \mathbf{R}(\mathbf{V},\nabla_{\mathbf{V}}\mathbf{Z})|_{r=\xi_3}\, r. \tag{33}$$

Applying the mean value theorem again to the first factor in the last term and using that $\mathbf{R}(\mathbf{V},\nabla_{\mathbf{V}}\mathbf{Z})|_p = \mathbf{R}(\mathbf{X},\mathbf{Y})|_p$ and $\nabla_V^2 Z = R(V,Z)V$ gives

$$\begin{aligned}\mathbf{R}(\mathbf{V},\mathbf{Z}) = {}& (\nabla_{\mathbf{V}}\mathbf{R})(\mathbf{V},\mathbf{Z})|_{r=\xi_3}\, r + \mathbf{R}(\mathbf{X},\mathbf{Y})|_p\, r \\ & + (\nabla_{\mathbf{V}}\mathbf{R})(\mathbf{V},\nabla_{\mathbf{V}}\mathbf{Z})|_{r=\xi_4}\,\xi_3\, r + \mathbf{R}\big(\mathbf{V},\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}\big)|_{r=\xi_4}\,\xi_3\, r\end{aligned} \tag{34}$$

for some $\xi_4 \in [0,\xi_3]$. Thus from (27) and (28),

$$\|\mathbf{R}(\mathbf{V},\mathbf{Z}) - \mathbf{R}(\mathbf{X},\mathbf{Y})|_p\, r\| < \frac{4001}{1999}\|\nabla\mathbf{R}\|_{D_l}\, r^2 + \frac{2000}{1999}\|\mathbf{R}\|_{D_l}^2\, r^3. \tag{35}$$

Integrating and using condition (I) and (II) along with $\alpha < 1$ we get

$$\big\|\lambda + \pi l^2\,\mathbf{R}(\mathbf{X},\mathbf{Y})|_p\big\| < 10^{-6}\alpha^2. \tag{36}$$

Adding (31) and (36) and applying Corollary 1 gives

$$\big\|\Lambda - \delta + \pi l^2\,\mathbf{R}(\mathbf{X},\mathbf{Y})|_p\big\| < 10^{-5}\alpha^2, \tag{37}$$

and we have established the first part of the lemma. The b-length of γ is given by

$$l(\gamma, E) = l(\rho_0, E) + l(o_l, E) + l(\rho_0, E\Lambda). \tag{38}$$

Now ρ_0 is a geodesic with $|\dot{\rho}_0| = |\mathbf{V}| = 1$, so the first and third terms are

$$l(\rho_0, E) = l \tag{39}$$

and

$$l(\rho_0, E\Lambda) \leq l\|\Lambda\| \leq l\exp(\|\lambda\|) < 1.1\,l \tag{40}$$

by (30). The second term is

$$l(o_l, E) = \int_0^{2\pi} |\mathbf{L}^{-1}\mathbf{Z}|_{r=l}\,\mathrm{d}\theta \leq \int_0^{2\pi} |\mathbf{Z}|\|\mathbf{L}\|_{r=l}\,\mathrm{d}\theta \tag{41}$$

since the norm of a Lorentz transformation equals the norm of its inverse. But $\mathbf{L}$ is given by Lemma 1, and applying (27), condition (I) and $\alpha < 1$ we get

$$\|\mathbf{L}\|_{r=l} \leq \exp\Big(\frac{1000}{1999} l^2 \|\mathbf{R}\|_{D_l}\,\theta\Big) < \exp\Big(\frac{\theta}{1999}\Big), \tag{42}$$

so using (27) again gives

$$l(o_l, E) < \frac{2000}{1999}\,l \int_0^{2\pi} \exp\Big(\frac{\theta}{1999}\Big)\,\mathrm{d}\theta < 6.3\,l. \tag{43}$$

Adding (39), (40) and (43) together we get the desired bound on $l(\gamma, E)$. □

Note. In [3], parallel propagation around a small square starting at one of the corners is investigated. The central result is Lemma 2.2.1, where the conditions $l^2\|\mathbf{R}\| < \alpha/28$ and $l\|\nabla\mathbf{R}\| < \|\mathbf{R}\|/20$ are used to establish

$$\|\Lambda - \delta - l^2\,\mathbf{R}(\mathbf{X}, \mathbf{Y})|_p\| < 6\alpha^2. \tag{44}$$

An explicit calculation in FLRW space-time illustrates that this is impossible without using a stronger condition on $\|\nabla\mathbf{R}\|$. Apart from typographical errors, the main problem seems to be in the argument at the top of page 24 of [3]. It is possible to obtain an estimate of order α^2 for parallel propagation around a circle with the starting point at the centre with a bound of order α on $\|\nabla\mathbf{R}\|$, by modifying the argument in our Lemma 2. The idea is to use a second order expansion of $\mathbf{R}(\mathbf{V}, \mathbf{Z})$ and then a symmetry argument to get rid of the first $\nabla\mathbf{R}$ term. However, the penalty for the weaker condition on $\nabla\mathbf{R}$ is that a condition on $\|\nabla^2\mathbf{R}\|$ of order α^2 has to be imposed. For our purpose, condition (II) is sufficient.

4 Generating Lorentz transformations

Using the approximation from Lemma 2, we can construct a loop generating a given Lorentz transformation exactly, provided that the transformation is sufficiently close to the identity. The idea is to generate a sequence of approximate transformations by parallel propagation along the boundaries of a sequence of appropriately constructed circles, applying Lemma 2 at each stage. First we construct the approximate curves to be used as building blocks for the final curve.

Note. To ensure the existence of the disks used to generate the curves, we need to avoid the situation where one of the radial curves cannot be continued because it runs into a singularity. If we restrict attention to a subset $\mathcal{U}$ of OM with compact closure, this can only happen if $\mathcal{U}$ contains a trapped inextendible incomplete curve [13]. This is avoided if we assume that the closure of $\mathcal{U}$ in $\overline{OM}$ is compact and contained in OM.

Lemma 3. *Let $\lambda \in \mathfrak{l}$, $E \in OM$ and $p = \pi(E)$ be given and suppose that there is a bivector $\mathbf{W}$ such that $\mathbf{R}_p(\mathbf{W}) = \lambda$, where $\mathbf{R}_p$ is the Riemann tensor in the frame E at p. Put*

$$\mathcal{U} := \{F; d(E,F) < 22\|\mathbf{W}\|^{1/2}\} \tag{45}$$

and $\|\cdot\|_{\mathcal{U}} := \sup_{\mathcal{U}} \|\cdot\|$, and let $\underline{\mathbf{L}} := \exp \lambda$. Also, assume that $\|\mathbf{W}\|$ is sufficiently small for the closure of $\mathcal{U}$ in $\overline{OM}$ to be compact and contained in OM. If

$$\|\mathbf{W}\| < (\pi/4000)\,\|\mathbf{R}\|_{\mathcal{U}}^{-1} \tag{I}$$

$$\|\mathbf{W}\| < (\pi/40000)\,\|\nabla\mathbf{R}\|_{\mathcal{U}}^{-2/3} \tag{II}$$

then there is a horizontal curve γ in $\mathcal{U}$ starting at E which generates a Lorentz transformation Λ with

$$\|\mathbf{L} - \Lambda\| < 10^{-3}\alpha^2, \tag{46}$$

where

$$\alpha < \max\left\{\frac{4000}{\pi}\|\mathbf{W}\|\|\mathbf{R}\|_{\mathcal{U}}, \left(\frac{40000}{\pi}\|\mathbf{W}\|\right)^{3/4}\|\nabla\mathbf{R}\|_{\mathcal{U}}^{1/2}\right\}, \tag{47}$$

and the b-length of γ is less than $22\|\mathbf{W}\|^{1/2}$.

Proof. We start by decomposing $\mathbf{W}$ as

$$\mathbf{W} = \mathbf{A}\cos\theta + *\mathbf{A}\sin\theta \tag{48}$$

where $\mathbf{A}$ and $*\mathbf{A}$ are dual independent simple bivectors. Inverting this relation and using that for any bivector $\mathbf{B}$,

$$\|*\mathbf{B}\| \le 2\sqrt{3}\|\mathbf{B}\|, \tag{49}$$

we get

$$\|\mathbf{A}\|, \|{*\mathbf{A}}\| < 4\|\mathbf{W}\|. \tag{50}$$

Define a disc by $f : D_{l_1} \to \pi(\mathcal{U})$, such that $|\mathbf{X}| = |\mathbf{Y}| = 1$, $\langle \mathbf{X}, \mathbf{Y} \rangle_{\mathbb{R}^4} = 0$ and $\pi l_1^2 \, \mathbf{X} \wedge \mathbf{Y} = -\mathbf{A} \cos\theta$ at E, as in §3. Then (50) gives

$$l_1^2 < \frac{4}{\pi} \|\mathbf{W}\|, \tag{51}$$

so $f(D_{l_1}) \subset \pi(\mathcal{U})$. Put

$$\alpha := \max\left\{10^3\, l_1^2 \|\mathbf{R}\|_{\mathcal{U}}, 10^3\, l_1^{3/2} \|\nabla \mathbf{R}\|_{\mathcal{U}}^{1/2}\right\}. \tag{52}$$

Then condition (I) and (II) give $\alpha < 1$ so Lemma 2 applies.

From Lemma 2 we have a loop γ_1 at p and a Lorentz transformation Λ_1 generated by parallel propagation around γ_1. Replacing $A\cos\theta$ and l_1 with $*A\sin\theta$ and l_2 and repeating the above procedure we get another loop γ_2 at p which generates a Lorentz transformation Λ_2. Put

$$\mathbf{Z}_1 := \Lambda_1 - \delta - \mathbf{R}_p(\mathbf{A}\cos\theta) \tag{53}$$

and

$$\mathbf{Z}_2 := \Lambda_2 - \delta - \mathbf{R}_p(*\mathbf{A}\sin\theta). \tag{54}$$

From Lemma 2 we know that

$$\|\mathbf{Z}_1\|, \|\mathbf{Z}_2\| < 10^{-5}\alpha^2. \tag{55}$$

Let $\Lambda = \Lambda_1\Lambda_2$. Then Λ is generated by parallel propagation around the concatenation γ of γ_1 and γ_2, and we may write

$$\begin{aligned}\Lambda - \mathbf{L} = \mathbf{Z}_1\big(\mathbf{Z}_2 + \delta + \mathbf{R}_p(*\mathbf{A}\sin\theta)\big) + \big(\delta + \mathbf{R}_p(\mathbf{A}\cos\theta)\big)\mathbf{Z}_2 \\ + \mathbf{R}_p(\mathbf{A}\cos\theta)\mathbf{R}_p(*\mathbf{A}\sin\theta) - \sum_{k=2}^{\infty} \frac{\lambda^k}{k!}.\end{aligned} \tag{56}$$

Using first (55) and (50) and then condition (I) we get

$$\|\Lambda_1\| = \|\mathbf{Z}_1 + \delta + \mathbf{R}_p(\mathbf{A}\cos\theta)\| < 10^{-5}\alpha^2 + 1 + 4\|\mathbf{R}_p\|\|\mathbf{W}\| < 1.01 \tag{57}$$

and similarly

$$\|\Lambda_2\| = \|\mathbf{Z}_2 + \delta + \mathbf{R}_p(*\mathbf{A}\sin\theta)\| < 1.01 \tag{58}$$

and

$$\|\delta + \mathbf{R}_p(\mathbf{A}\cos\theta)\| < 1.01. \tag{59}$$

Inserting (58) and (59) into (56) and using that

$$\|\lambda\| \leq \|\mathbf{R}_p\|\|\mathbf{W}\| < \frac{\pi}{4000} \tag{60}$$

and

$$\|\mathbf{W}\| < \pi(l_1^2 + l_2^2) \tag{61}$$

and condition (I) gives

$$\|\Lambda - \mathbf{L}\| < 2.02 \cdot 10^{-5}\alpha^2 + 16\|\mathbf{R}_p\|^2\|\mathbf{W}\|^2 + \frac{\|\lambda\|^2}{2(1 - \|\lambda\|)} < 10^{-3}\alpha^2. \tag{62}$$

The length of γ is the sum of the lengths of γ_1 and γ_2. From Lemma 2 and (51) we find that

$$l(\gamma_1, E) < 9l_1 < 11\|\mathbf{W}\|^{1/2}. \tag{63}$$

The same holds for γ_2 except that we have to correct for the starting frame being $E\Lambda_1$ instead of E. From (57),

$$l(\gamma_2, E\Lambda_1) < 9l_2\|\Lambda_1\| < 11\|\mathbf{W}\|^{1/2}, \tag{64}$$

and thus

$$l(\gamma, E) < 22\|\mathbf{W}\|^{1/2}. \tag{65}$$

□

The Riemann tensor in a given frame can be viewed as a map from the space of bivectors to the Lie algebra $\mathfrak{l}$. We use the norm

$$\|\mathbf{W}\| := 2 \sup_{|\mathbf{X}|=|\mathbf{Y}|=1} |W^{ij}X_iY_j| \tag{66}$$

for the bivectors, so that the mapping norm

$$\|\mathbf{R}\| := \sup_{\|\mathbf{W}\|=1} \|\mathbf{R}(\mathbf{W})\| \tag{67}$$

agrees with the previously defined tensor norm (4). We now concentrate on the case when the Riemann tensor in the frame E is invertible (the frame is of course not essential here since invertibility in one frame is equivalent to invertibility in any frame). Note that if the Riemann tensor is invertible at a point $F \in OM$, the image of the space of bivectors is the whole Lorentz group $\mathcal{L}$, so by the standard holonomy theory the infinitesimal holonomy group is the whole of $\mathcal{L}$. Thus the length estimate is the important result here. The idea is to piece the curves from Lemma 3 together to generate a sufficiently small Lorentz transformation exactly.

Lemma 4. *Let $\lambda \in \mathfrak{l}$, $E \in OM$ and $p := \pi(E)$ be given and suppose that $\mathbf{R}_p$, the Riemann tensor in the frame E at p, is invertible. Let $\mathbf{R}_p^{-1}$ be the inverse and put*

$$\mathcal{U} := \{F; d(E,F) < 24\|\mathbf{R}_p^{-1}\|^{1/2}\|\lambda\|^{1/2}\}. \tag{68}$$

and $\|\cdot\|_{\mathcal{U}} := \sup_{\mathcal{U}} \|\cdot\|$. If the closure of $\mathcal{U}$ in $\overline{OM}$ is compact and contained in OM and

$$\|\lambda\| < 10^{-6}\, \|\mathbf{R}_p^{-1}\|^{-2}\|\mathbf{R}\|_{\mathcal{U}}^{-2} \tag{I}$$

$$\|\lambda\| < 10^{-12}\, \|\mathbf{R}_p^{-1}\|^{-3}\|\nabla\mathbf{R}\|_{\mathcal{U}}^{-2} \tag{II}$$

then there is a horizontal curve in $\mathcal{U}$ starting from E and ending at $E \exp \lambda$, piecewise smooth except possibly at the endpoint, of b-length less than

$$24\|\mathbf{R}_p^{-1}\|^{1/2}\|\lambda\|^{1/2}. \tag{69}$$

Proof. Let $\mathbf{L} := \exp \lambda$. To construct the first square, put $\mathbf{W} := \mathbf{R}_p^{-1}(\lambda)$. Since

$$\|\mathbf{R}_p^{-1}\|\|\mathbf{R}\|_{\mathcal{U}} \geq \|\mathbf{R}_p^{-1}\|\|\mathbf{R}_p\| \geq 1, \tag{70}$$

condition (I) gives

$$\|\lambda\| < 10^{-6}. \tag{71}$$

Applying condition (I) to the first factor of $\|\lambda\|^2$ and (71) to the second factor and then taking the square root gives that condition (I) of Lemma 3 is fulfilled. Similarly, applying condition (II) to the first factor of $\|\lambda\|^3$, (71) to the other two factors and taking the third root gives that condition (II) of Lemma 3 is fulfilled. Thus Lemma 3 applies and we have a loop γ_1 which generates a first approximation $\mathbf{L}_1$ to $\mathbf{L}$. Also,

$$\begin{aligned}\alpha^2 < \|\lambda\| \max\Big\{ &\Big(\frac{4000}{\pi}\Big)^2 \|\lambda\|\|\mathbf{R}_p^{-1}\|^2\|\mathbf{R}\|_{\mathcal{U}}^2, \\ &\Big(\frac{40000}{\pi}\Big)^{3/2}\big(\|\lambda\|\|\mathbf{R}_p^{-1}\|^3\|\nabla\mathbf{R}\|_{\mathcal{U}}^2\big)^{1/2}\Big\}\end{aligned} \tag{72}$$

so from condition (I) and (II),

$$\alpha^2 < 2\|\lambda\|, \tag{73}$$

and Lemma 3 gives

$$\|\mathbf{L} - \mathbf{L}_1\| < \frac{1}{500}\|\lambda\|. \tag{74}$$

Next we repeat the construction for the Lorentz transformation $\mathbf{L}_1^{-1}\mathbf{L}$. We first have to check that the conditions are satisfied. But from (57) and (58),

$$\|\mathbf{L}_1\| \leq \|\Lambda_1\|\|\Lambda_2\| < 1.1, \tag{75}$$

and from (74) and the fact that the norm of a Lorentz transformation equals the norm of its inverse,

$$\|\mathbf{L}_1^{-1}\mathbf{L} - \delta\| < \|\mathbf{L}_1\|\|\mathbf{L} - \mathbf{L}_1\| < \frac{1}{450}\|\lambda\|. \tag{76}$$

It follows that we can write $\mathbf{L}_1^{-1}\mathbf{L} = \exp\lambda_2$ with

$$\|\lambda_2\| < \frac{450}{449}\|\mathbf{L}_1^{-1}\mathbf{L} - \delta\| < \frac{1}{449}\|\lambda\|. \tag{77}$$

Thus λ_2 satisfies the conditions as long as the generating curve stays in $\mathcal{U}$.

Repeating the above process we get a series of loops γ_k corresponding to a sequence λ_k of Lie algebra elements, generating Lorentz transformations $\mathbf{L}_k$. The products $\hat{\mathbf{L}}_k = \mathbf{L}_1\mathbf{L}_2 \ldots \mathbf{L}_k$ are generated by parallel propagation along the concatenation of the curves $\gamma_1, \gamma_2, \ldots, \gamma_k$, and

$$\|\hat{\mathbf{L}}_k - \mathbf{L}\| \le \|\hat{\mathbf{L}}_{k-1}\|\|\mathbf{L}_k - \hat{\mathbf{L}}_{k-1}^{-1}\mathbf{L}\| < 1.1^{k-1}\frac{1}{500}\|\lambda_k\| \tag{78}$$

from (74) and repeated application of (75). But (77) gives

$$\|\lambda_k\| < \Big(\frac{1}{449}\Big)^{k-1}\|\lambda\|, \tag{79}$$

so $\hat{\mathbf{L}}_k \to \mathbf{L}$ as $k \to \infty$. It remains to show that the resulting curve is contained in $\mathcal{U}$. From Lemma 3,

$$l(\gamma_1, E) < 22\|\mathbf{R}_p^{-1}\|^{1/2}\|\lambda\|^{1/2}. \tag{80}$$

For γ_k, we have to take into account that the starting point is $E\hat{\mathbf{L}}_{k-1}$ instead of E, so

$$\begin{aligned} l(\gamma_k, E\mathbf{L}_{k-1}) &< 22\|\mathbf{R}_p^{-1}\|^{1/2}\|\lambda_k\|^{1/2}\|\hat{\mathbf{L}}_{k-1}\| \\ &< 22\|\mathbf{R}_p^{-1}\|^{1/2}\Big(\frac{1}{449}\Big)^{(k-1)/2}\|\lambda\|^{1/2}\,1.1^{k-1} \end{aligned} \tag{81}$$

from (75) and (79). Summing over k we get the desired bound on the length, and it is evident that the generating curve stays in $\mathcal{U}$. $\square$

Note. The main difference between our Lemma 4 and Lemma 2.2.2 of [3] is that condition (I) involves the second power of $\mathbf{R}_p^{-1}$ and $\mathbf{R}$ instead of the first. This is needed to establish (72) which is essential for the construction of the sequence of circles to work. The corresponding equation at the bottom of page 26 in [3] is incorrect since there a bound on $\Gamma^2\|\lambda\|$ is needed, but the given conditions only provide a bound on $\Gamma\|\lambda\|$.

It is now a simple matter to generate arbitrary transformations by splitting them in a finite number of factors, sufficiently small for Lemma 4 to apply, and joining together the resulting curves. Note that we do not need to go through the approximation scheme in Lemma 4 more than once as is done in [3], since once we have a curve generating the first factor, we can translate it along the fibres to get curves generating the other factors.

Theorem 1. *Let $E \in OM$ with $p := \pi(E)$ and put*

$$\mathcal{U} := \{F \in OM; d(E, F) < \delta\} \tag{82}$$

for some $\delta > 0$, small enough for the closure of $\mathcal{U}$ in $\overline{OM}$ to be compact and contained in OM. Let $\mathbf{L} := \exp \lambda$ be a Lorentz transformation and suppose that $\mathbf{R}$ is invertible on $\mathcal{U}$. Then there is a horizontal curve γ in $\pi^{-1} \circ \pi(\mathcal{U})$ which generates $\mathbf{L}$ with

$$l(\gamma, E) < 24\|\mathbf{L}\| \|\mathbf{R}_p^{-1}\|^{1/2} \|\lambda\|^{1/2} n^{1/2}, \tag{83}$$

where

$$n := \left\lceil \|\lambda\| \max \left\{10^6 \|\mathbf{R}_p^{-1}\|^2 \|\mathbf{R}\|_{\mathcal{U}}^2, 10^{12} \|\mathbf{R}_p^{-1}\|^3 \|\nabla \mathbf{R}\|_{\mathcal{U}}^2, 24^2 \|\mathbf{R}_p^{-1}\| / \delta^2 \right\} \right\rceil. \tag{84}$$

Proof. We start by generating the Lorentz transformation $\mathbf{L}_1 := \exp(\lambda/n)$ where $n \in \mathbb{N}$ is chosen sufficiently large for Lemma 4 to hold on a subset of $\mathcal{U}$, which gives (84). By Lemma 4 there exists a horizontal curve γ_1 in $\mathcal{U}$ from E to $E_1 := E\mathbf{L}_1$. Let $\mathbf{L}_k := (\mathbf{L}_1)^k$ and $E_k := E\mathbf{L}_k$ for $k = 2, 3, \ldots, n$. Then $\gamma_k = \gamma_1 \mathbf{L}_{k-1}$ is a horizontal curve from E_{k-1} to E_k since the action of the Lorentz group preserves horizontal curves.

Let γ be the combined curve obtained by joining the curves γ_k in sequence. Then γ generates $\mathbf{L}_n = \mathbf{L}$ and since

$$l(\gamma_k) \leq \|\mathbf{L}_k\| \, l(\gamma_1) \leq \|\mathbf{L}\| \, l(\gamma_1) \tag{85}$$

and

$$l(\gamma_1) < 24 \|\mathbf{R}_p^{-1}\|^{1/2} \|\lambda/n\|^{1/2}, \tag{86}$$

the result follows. □

5 The singular holonomy group

We can now relate the structure of the singular holonomy group with the asymptotic behaviour of the Riemann tensor. First we need the following characterisation from [3].

Proposition 2. *Suppose that $\gamma : (0, 1] \to OM$ is a horizontal curve with $\gamma(0) = E \in \overline{\pi}^{-1}(p)$ and $p \in \partial M$. Then $\mathbf{L} \in \Phi^s_{OM}(E)$ if and only if there is a sequence t_i with $t_i \to 1$ and loops $\kappa_i : [0, 1] \to M$ such that*

$$\kappa_i(0) = \kappa_i(1) = \pi \circ \gamma(t_i), \tag{I}$$

$$\mathbf{L}_i \to \mathbf{L} \tag{II}$$

$$l(\kappa_i, \gamma(t_i)) \to 0 \tag{III}$$

where $\mathbf{L}_i$ are the Lorentz transformations obtained by parallel propagating $\gamma(t_i)$ around κ_i for each i.

We may use Proposition 2 to give an alternative definition of the singular holonomy group [2]. Let $\varphi_a(F)$ be the group of Lorentz transformations generated by parallel transport around loops κ at $\pi(F)$ with $l(\kappa, F) \leq a$. Then if $\gamma : (0,1] \to OM$ is a horizontal curve starting at $\gamma(0) = E \in \overline{\pi}^{-1}(p)$ with $p \in \partial M$,

$$\Phi^s_{OM}(E) := \bigcap_{a \in \mathbb{R}^+} \overline{\bigcup_{t \in (0,1]} \varphi_a(\gamma(t))}. \tag{87}$$

A nontrivial Φ^s_{OM} may have several causes. For example, the bounded part of the curvature may contribute as well as the unbounded part [2], and non-trivial topologies can generate discrete subgroups (see §7 below). In the following section we concentrate on using Lemma 4 to show how divergence of the Riemann tensor can cause total degeneracy.

6 Total Degeneracy

Combining Proposition 2 with Theorem 1 we get the following sufficient conditions for total degeneracy of a boundary fibre. In the rest of this section we will see that the conditions are indeed fulfilled in many interesting cases relevant to general relativity.

Theorem 2. *Suppose that $\gamma : (0,1] \to OM$ is a horizontal curve with $\gamma(0) = E \in \overline{\pi}^{-1}(p)$ and $p \in \partial M$, and that there are sequences $t_i \to 0$ and $\rho_i \to 0$ such that $\mathbf{R}$ is invertible on the balls $\mathcal{U}_i := B_{\rho_i}(\gamma(t_i))$. If the closure of each $\mathcal{U}_i$ in $\overline{OM}$ is compact and contained in OM and $\|\mathbf{R}_i^{-1}\|^3 \|\mathbf{R}\|^2_{\mathcal{U}_i}$, $\|\mathbf{R}_i^{-1}\|^2 \|\nabla \mathbf{R}\|_{\mathcal{U}_i}$ and $\|\mathbf{R}_i^{-1}\|/\rho_i$ tend to 0 as $t_i \to 0$, then $\Phi^s_{OM}(E) = \mathcal{L}$.*

Note that invertibility of the Riemann tensor means that it is injective, i.e. there are no 2-planes on which $\mathbf{R}$ vanishes, and surjective, i.e. there is no subspace of the Lie algebra unaffected by curvature. If $\mathbf{R}$ is invertible,

$$\|\mathbf{R}^{-1}\| = \sup_{\lambda} \frac{\|\mathbf{R}^{-1}(\lambda)\|}{\|\lambda\|} = \sup_{\mathbf{W}} \frac{\|\mathbf{W}\|}{\|\mathbf{R}(\mathbf{W})\|} = \Big(\inf_{\|\mathbf{W}\|=1} \|\mathbf{R}(\mathbf{W})\| \Big)^{-1}, \tag{88}$$

so $\|\mathbf{R}^{-1}\| \to 0$ if and only if $\|\mathbf{R}(\mathbf{W})\|$ diverges for all bivectors $\mathbf{W}$. In other words, $\|\mathbf{R}^{-1}\| \to 0$ if and only if, for all index pairs k and l, there are two indices i and j such that the frame component $\mathbf{R}^i{}_{jkl}$ diverges. This could happen if all sectional curvatures diverge, for example.

We are now able to show that the boundary fibres are totally degenerate in many cases. We will employ the following procedure. Let $\gamma : I \to M$ be a curve with an endpoint $p \in \partial M$, and let E be a pseudo-orthonormal frame field on (a subset of) M. Using Cartan's equations we find the rotation coefficients and the Riemann tensor components in the frame E. We may then write down and solve the parallel propagation equations for a frame F along γ. The tricky part is finding a sequence of parameter values t_i along with suitable ρ_i-balls $\mathcal{U}_i$ and bounds on $\|\mathbf{R}\|_{\mathcal{U}_i}$ and $\|\nabla \mathbf{R}\|_{\mathcal{U}_i}$. To this end, we need to explore the connection between the b-distance and Lorentz transformations.

Lemma 5. *Let $p \in M$ and $\mathcal{V} \subseteq B_\rho(p, E_p) \subset OM$, and suppose that E_p can be extended to a frame field E on $\mathcal{V}$. Put $\|\Gamma\|_{\pi(\mathcal{V})} := \sup_{\pi(\mathcal{V})}\|\Gamma\|$, where Γ is the array of the rotation coefficients in the frame E, and $K := \max\{\|\Gamma\|_{\pi(\mathcal{V})}, 1\}$. If $\rho \le 1/4K$ then all frames in $\mathcal{V}$ can be expressed as $E\mathbf{L}$ with $\|\mathbf{L}\| < 2$.*

Proof. Let $\kappa : [0, \rho] \to \mathcal{V}$ be a curve in $\mathcal{V}$, parameterised by b-length, with $\kappa(0) = (p, E)$. Let $\dot\kappa$ be the tangent vector of κ, and let V be the tangent vector of $\pi \circ \kappa$ with components $\mathbf{V}$ in the fixed frame E. Also, let the frame F of κ be given by $F = E\mathbf{L}$. We want to show that $\|\mathbf{L}\| < 2$.

From [8], the fundamental 1-form θ at $\kappa(s)$ is given by $F^{-1} \circ \pi_*$, where F is regarded as a map $\mathbb{R}^4 \to T_{\pi\circ\kappa}M$, so

$$\theta(\dot\kappa) = \mathbf{L}^{-1}\mathbf{V}. \tag{89}$$

Next, the connection form ω is given by

$$\varphi\big(\omega(\dot\kappa)\big) = \mathrm{ver}\,\dot\kappa, \tag{90}$$

where φ is the canonical isomorphism from $\mathfrak{l}$ to the vertical subspace of $T_{\kappa(s)}OM$, and $\mathrm{ver}\,\dot\kappa$ denotes the vertical component of $\dot\kappa$ [8]. By definition, if $a \in \mathfrak{l}$ and $A(t)$ is any curve in $\mathcal{L}$ with $A(0) = \delta$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} A = a$, then

$$\varphi(a) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} R_{A(t)}F = Fa \tag{91}$$

at F. The vertical component of $\dot\kappa$ is given by

$$\nabla_V F = (\nabla_V E)\mathbf{L} + E\dot{\mathbf{L}} = F\mathbf{L}^{-1}(\Gamma\mathbf{V}\mathbf{L} + \dot{\mathbf{L}}), \tag{92}$$

where $\Gamma\mathbf{V}\mathbf{L}$ is the matrix with components $\Gamma^i{}_{kl}\mathbf{V}^k\mathbf{L}^l{}_j$ and $\Gamma^i{}_{kl}$ are the rotation coefficients of the frame E. Combining (90), (91) and (92) gives

$$\omega(\dot\kappa) = \mathbf{L}^{-1}(\Gamma\mathbf{V}\mathbf{L} + \dot{\mathbf{L}}). \tag{93}$$

Since κ is parameterised by b-length,

$$|\theta(\dot\kappa)|^2 + \|\omega(\dot\kappa)\|^2 = 1, \tag{94}$$

so from (89),

$$|\mathbf{V}| \le \|\mathbf{L}\||\theta(\dot\kappa)| \le \|\mathbf{L}\| \tag{95}$$

and from (93),

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\|\mathbf{L}\|\right| \le \|\mathbf{L}\|\|\omega(\dot\kappa)\| + \|\Gamma\||\mathbf{V}|\|\mathbf{L}\| \le K\|\mathbf{L}\|^2 + \|\mathbf{L}\|. \tag{96}$$

Put $u := K\|\mathbf{L}\|$. Then

$$\frac{\dot u}{u^2 + u} \le 1, \tag{97}$$

and integration gives

$$u \leq \frac{K}{(K+1)e^{-s} - K} \tag{98}$$

since $\|\mathbf{L}\| = 1$ at $s = 0$. Thus

$$\|\mathbf{L}\| \leq \big((K+1)e^{-s} - K\big)^{-1}, \tag{99}$$

and the result follows from $s \leq 1/4K$ and $K \geq 1$. □

Note. (96) corresponds to the differential equation on page 42 of [3], except that there the last term is 1 instead of $\|\mathbf{L}\|$ which is incorrect.

6.1 FLRW space-times

Let (M, g) be a Robertson-Walker space-time, i.e. $M = (0, \tau) \times \Sigma$ and g is defined by the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\sigma^2 \tag{100}$$

such that $(\Sigma, \mathrm{d}\sigma^2)$ is a homogeneous space (see eg. [6, 10, 3]). The scale function $a(t)$ is determined from the chosen matter model via the field equations. For a Friedman big bang model, $a(t) \to 0$ as $t \to 0$, corresponding to a curvature singularity at $t = 0$.

Let γ be a curve in M with constant projection $x \in \Sigma$, parameterised by t. Then γ starts at the singularity at $t = 0$. Choose the pseudo-orthonormal frame field E on (a subset of) M as

$$E_0 := \frac{\partial}{\partial t} \qquad \text{and} \qquad E_\alpha := a(t)^{-1} \tilde{E}_\alpha \tag{101}$$

where $\tilde{E}$ is an orthonormal frame field on the Riemannian manifold $(\Sigma, \mathrm{d}\sigma^2)$. Note that $\tilde{E}$ may be defined only on a neighbourhood of x if $(\Sigma, \mathrm{d}\sigma^2)$ does not admit a global parallelisation. Here greek indices $\alpha, \beta, \ldots$ refer to spatial components and have values in $\{1, 2, 3\}$. Write θ for the cotangent frame field dual to E, i.e. θ is the fundamental 1-form restricted to the section of OM defined by E. From Cartan's equations, the nonvanishing connection and curvature form components are

$$\begin{aligned} \omega^0{}_\alpha &= \omega^\alpha{}_0 = \dot{a} a^{-1}\, \theta^\alpha \\ \omega^\alpha{}_\beta &= -\omega^\beta{}_\alpha = a^{-1}\, \tilde{\Gamma}^\alpha{}_{\mu\beta}\, \theta^\mu \end{aligned} \tag{102}$$

and

$$\begin{aligned} \Omega^0{}_\alpha &= \Omega^\alpha{}_0 = \ddot{a} a^{-1}\, \theta^0 \wedge \theta^\alpha \\ \Omega^\alpha{}_\beta &= -\Omega^\beta{}_\alpha = (a^{-2}\, \tilde{\mathbf{R}}^\alpha{}_{\beta\mu\nu} + \dot{a}^2 a^{-2}\, \delta^\alpha{}_\mu \delta_{\beta\nu})\, \theta^\mu \wedge \theta^\nu \end{aligned} \tag{103}$$

where a dot denotes differentiation with respect to t and $\tilde{\Gamma}^{\alpha}{}_{\delta\beta}$ and $\tilde{\mathbf{R}}^{\alpha}{}_{\beta\mu\nu}$ are the rotation coefficients and the Riemann tensor components, respectively, of $(\Sigma, \mathrm{d}\sigma^2)$ in the frame $\tilde{E}$.

Solving the parallel propagation equations we find that E is parallel along γ. To study the asymptotic behaviour we consider the case $a(t) = t^c$ for a constant $c \in (0, 1)$. Then there are positive constants N_1 and N_2 such that

$$\|\mathbf{R}\| < N_1 \max\{t^{-2}, t^{-2c}\|\tilde{\mathbf{R}}\|\} \tag{104}$$

and

$$\|\nabla\mathbf{R}\| < N_2 \max\{t^{-3}, t^{-2c-1}\|\tilde{\mathbf{R}}\|, t^{-3c}\|\nabla\tilde{\mathbf{R}}\|\} \tag{105}$$

in the frame E. Moreover, $\mathbf{R}$ is invertible on γ and

$$\|\mathbf{R}^{-1}\| < N_3\, t^2 \tag{106}$$

for some positive constant N_3, so $\|\mathbf{R}^{-1}\| \to 0$ as $t \to 0$. Pick a sequence $t_i \to 0$ and let $\rho_i := t_i/8$ and $\mathcal{U}_i := B_{\rho_i}\big(\gamma(t_i)\big)$. Let $\mathcal{S}$ be a neighbourhood of x in Σ such that $\|\tilde{\Gamma}\|$, $\|\tilde{\mathbf{R}}\|$ and $\|\nabla\tilde{\mathbf{R}}\|$ are bounded on $\mathcal{S}$. Put $\mathcal{V}_i := \mathcal{U}_i \cap \mathcal{K}_i$, where $\mathcal{K}_i := \pi^{-1}([t_i/2, 3t_i/2] \times \mathcal{S})$. Then for small enough t_i,

$$1 < \|\Gamma\|_{\mathcal{V}_i} \leq 2ct_i^{-1} = K_i \tag{107}$$

and since $c < 1$, $\rho_i < 1/4K_i$. Thus Lemma 5 gives $\|\mathbf{L}\| < 2$ on $\mathcal{V}_i$. If κ is a curve in $\mathcal{V}_i$ with $\kappa(0) = \gamma(t_i)$ and $l(\kappa) \leq \rho_i$, the t-coordinate satisfies

$$|t - t_i| = \left| \int_0^s \big(\mathbf{L}\theta(\dot{\kappa})\big)^0 \,\mathrm{d}s \right| \leq \|\mathbf{L}\| l(\kappa) < \frac{t_i}{4} \tag{108}$$

on κ. Let $\tilde{\kappa}$ be the projection of $\pi \circ \kappa$ to Σ. Since $\tilde{E}$ is an orthonormal frame, the metric length of $\tilde{\kappa}$ in $(\Sigma, \mathrm{d}\sigma^2)$ can be estimated by

$$l_\sigma(\tilde{\kappa}) \leq \int_0^s a^{-1}\|\mathbf{L}\||\theta(\dot{\kappa})| \,\mathrm{d}s < 2^{c-2}\, t_i^{1-c} \tag{109}$$

which tends to 0 as $t_i \to 0$. But then $\mathcal{U}_i$ must be contained in $\mathcal{K}_i$ for small enough t_i, so the t-coordinate must be greater than $t_i/2$ on the whole of $\mathcal{U}_i$.

Thus $\|\mathbf{R}_i^{-1}\|^3\|\mathbf{R}\|^2_{\mathcal{U}_i}$, $\|\mathbf{R}_i^{-1}\|^2\|\nabla\mathbf{R}\|_{\mathcal{U}_i}$ and $\|\mathbf{R}_i^{-1}\|/\rho_i$ all tend to 0 as $t_i \to 0$, so by Theorem 2 the fibre over $\gamma(0)$ is totally degenerate. Note that in [3], a similar result is given for $2/3 < c < 1$. The reason for the restriction on c is that Clarke uses a bound on $\|\mathbf{R}^{-1}\|$ of order t^{2c}, while $\|\mathbf{R}^{-1}\|$ is actually of order t^2 for small enough t.

6.2 Kasner space-times

To illustrate that the fibre degeneracy is not an artefact of isotropy we repeat the calculations for the anisotropic Kasner space-times (see e.g. [10]). Let $M := I \times \Sigma$ with metric g given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2p_x}\mathrm{d}x^2 + t^{2p_y}\mathrm{d}y^2 + t^{2p_z}\mathrm{d}z^2, \tag{110}$$

where (x, y, z) are coordinates on Σ and the constants p_x, p_y and p_z satisfy

$$p_x + p_y + p_z = 1 \qquad \text{and} \qquad p_x^2 + p_y^2 + p_z^2 = 1. \tag{111}$$

We exclude the special case when $p_x = p_y = 0$, $p_z = 1$ (including permutations of x, y and z) which corresponds to one half of Minkowski space. For all other parameter values, there is a curvature singularity at $t = 0$. Let γ be a curve with constant x, y and z, starting at the singularity and parameterised by t. Choosing a pseudo-orthonormal frame field E as

$$E_0 := \frac{\partial}{\partial t}, \quad E_1 := t^{-p_x}\frac{\partial}{\partial x}, \quad E_2 := t^{-p_y}\frac{\partial}{\partial y} \quad \text{and} \quad E_3 := t^{-p_z}\frac{\partial}{\partial z} \tag{112}$$

we find again that E is parallel propagated along γ, that $\mathbf{R}$ is invertible, and that $\|\mathbf{R}\| < N_1\, t^{-2}$, $\|\nabla\mathbf{R}\| < N_2\, t^{-3}$ and $\|\mathbf{R}^{-1}\| < N_3\, t^2$ for some constants N_1, N_2 and N_3. Put $p := \max\{|p_x|, |p_y|, |p_z|\}$. Then $\|\Gamma\| = pt^{-1}$, and an argument similar to that in §6.1 gives that the boundary fibre is totally degenerate.

6.3 Schwarzschild space-time

Let (M, g) be given by

$$\mathrm{d}s^2 = b(r)^{-2}\mathrm{d}t^2 - b(r)^2\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\phi^2) \tag{113}$$

with $t \in \mathbb{R}$, $r \in (0, 2m)$, $\vartheta \in [0, \pi]$, $\phi \in [0, 2\pi)$, and

$$b(r) := \Big(\frac{2m}{r} - 1\Big)^{-1/2} \tag{114}$$

(see e.g. [6, 10]). Choose E as

$$E_0 := b^{-1}\frac{\partial}{\partial r}, \quad E_1 := b\frac{\partial}{\partial t}, \quad E_2 := r^{-1}\frac{\partial}{\partial\vartheta} \quad \text{and} \quad E_3 := (r\sin\vartheta)^{-1}\frac{\partial}{\partial\phi} \tag{115}$$

and let the corresponding cotangent frame be θ. The connection form is

$$\begin{aligned}
&\omega^0{}_1 = \omega^1{}_0 = -mbr^{-2}\,\theta^1 &&\omega^0{}_2 = \omega^2{}_0 = b^{-1}r^{-1}\,\theta^2\\
&\omega^0{}_3 = \omega^3{}_0 = b^{-1}r^{-1}\,\theta^3 &&\omega^2{}_3 = -\omega^3{}_2 = -r^{-1}\cot\vartheta\,\theta^3
\end{aligned} \tag{116}$$

and the curvature form is

$$\begin{aligned}
&\Omega^0{}_1 = \Omega^1{}_0 = 2mr^{-3}\,\theta^0\wedge\theta^1 &&\Omega^0{}_2 = \Omega^2{}_0 = -mr^{-3}\,\theta^0\wedge\theta^2\\
&\Omega^0{}_3 = \Omega^3{}_0 = -mr^{-3}\,\theta^0\wedge\theta^3 &&\Omega^1{}_2 = -\Omega^2{}_1 = -mr^{-3}\,\theta^1\wedge\theta^2\\
&\Omega^1{}_3 = -\Omega^3{}_1 = -mr^{-3}\,\theta^1\wedge\theta^3 &&\Omega^2{}_3 = -\Omega^3{}_2 = 2mr^{-3}\,\theta^2\wedge\theta^3.
\end{aligned} \tag{117}$$

Thus there are positive constants N_1 and N_2 such that $\|\mathbf{R}\| < N_1\, r^{-3}$ and $\|\nabla\mathbf{R}\| < N_2\, r^{-9/2}$ in the frame E.

Let γ be a radial curve parameterised by r with $\vartheta = \pi/2$, $\phi = 0$ and $t = t_0$. Then E is parallel on γ, $\mathbf{R}$ is invertible, and $\|\mathbf{R}^{-1}\| < r^3/m$ along γ. If ϑ is bounded away from 0 and π,

$$\|\Gamma\| \leq \sqrt{2m} r^{-3/2} \tag{118}$$

for small r. Choosing a sequence $r_i \to 0$ and

$$\rho_i := \frac{r_i^{3/2}}{16\sqrt{m}}, \tag{119}$$

an argument similar to that in §6.1 gives $\|\mathbf{L}\| < 2$ and $r > r_i/2$ on each $\mathcal{U}_i := B_{\rho_i}\big(\gamma(r_i)\big)$ for small enough r_i. Thus the conditions of Theorem 2 are fulfilled, so the boundary fibre is totally degenerate.

6.4 Reissner-Nordström space-time

Let (M, g) be given by

$$\mathrm{d}s^2 = -b(r)^{-2}\mathrm{d}t^2 - b(r)^2\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\phi^2) \tag{120}$$

with $t \in \mathbb{R}$, $r \in (0, r_-)$, $\vartheta \in [0, \pi]$ and $\phi \in [0, 2\pi)$, and

$$b(r) := \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1/2} \tag{121}$$

(see e.g. [6, 10]). Degeneracy of the boundary fibre follows directly by generalising the argument in §6.3, with $\rho_i := r_i^2/32|e|$, $\|\mathbf{R}\| < N_1\, r^{-4}$, $\|\nabla\mathbf{R}\| < N_2\, r^{-6}$ and $\|\mathbf{R}^{-1}\| < N_3\, r^4$. Note that the timelike nature of the singularity does not affect the argument.

6.5 Tolman-Bondi space-time

The metric for the spherically symmetric Tolman-Bondi space-time (M, g) is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2\omega}\mathrm{d}r^2 + R^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\phi^2) \tag{122}$$

where $\omega := \omega(t, r)$ and $R := R(t, r) > 0$ [11]. If the energy momentum tensor is taken to be of dust form,

$$T := \epsilon(t, r)\,\frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t}, \tag{123}$$

the equations for ω and R are

$$\frac{1}{2}\dot{R}^2 - \frac{m}{R} = \frac{1}{2}(W^2 - 1) \tag{124}$$

$$R' = We^{\omega} \tag{125}$$

$$\epsilon = \frac{r^2\rho}{R^2 R'} \tag{126}$$

where $W := W(r)$, $\rho(r) := \epsilon(0,r)$, dots and primes denote partial derivatives with respect to t and r respectively, and

$$m(r) := 4\pi \int_0^r \rho r^2 \,\mathrm{d}r. \tag{127}$$

Here r is rescaled such that $r := R(0,r)$ and ω, R and ϵ are assumed to be smooth functions of t and r. For physical reasons, we require that $\epsilon(t,r) \geq 0$ and $\epsilon(t,0) > 0$. Put

$$E(r) := \frac{1}{2}\big(W^2(r) - 1\big) \tag{128}$$

and let

$$a(r) := \frac{3m(r)}{4\pi r^3} \tag{129}$$

and

$$p(r) := -\frac{E(r)R(0,r)}{m(r)}. \tag{130}$$

It can be shown that both a and p extend to smooth even functions of r on $\mathbb{R}$, with $a(r) > 0$ and $p(r) \leq 1$.

Choose a pseudo-orthonormal frame E with cotangent frame θ according to

$$E_0 := \frac{\partial}{\partial t}, \quad E_1 := \frac{W}{R'}\frac{\partial}{\partial r}, \quad E_2 := R^{-1}\frac{\partial}{\partial \vartheta}, \quad \text{and} \quad E_3 := (R\sin\vartheta)^{-1}\frac{\partial}{\partial \phi}. \tag{131}$$

Then the connection form is

$$\begin{aligned}
&\omega^0{}_1 = \omega^1{}_0 = \frac{\dot{R}'}{R'}\,\theta^1 && \omega^0{}_2 = \omega^2{}_0 = \frac{\dot{R}}{R}\,\theta^2 && \omega^0{}_3 = \omega^3{}_0 = \frac{\dot{R}}{R}\,\theta^3 \\
&\omega^1{}_2 = -\omega^2{}_1 = -\frac{W}{R}\,\theta^2 && \omega^1{}_3 = -\omega^3{}_1 = -\frac{W}{R}\,\theta^3 && \omega^2{}_3 = -\omega^3{}_2 = -\frac{1}{R}\cot\vartheta\,\theta^3
\end{aligned} \tag{132}$$

and the curvature form is

$$\begin{aligned}
&\Omega^0{}_1 = \Omega^1{}_0 = 2mR^{-3}\,\theta^0 \wedge \theta^1 && \Omega^0{}_2 = \Omega^2{}_0 = -mR^{-3}\,\theta^0 \wedge \theta^2 \\
&\Omega^0{}_3 = \Omega^3{}_0 = -mR^{-3}\,\theta^0 \wedge \theta^3 && \Omega^1{}_2 = -\Omega^2{}_1 = \Big(\frac{m'}{R'R^2} - \frac{m}{R^3}\Big)\,\theta^1 \wedge \theta^2 \\
&\Omega^1{}_3 = -\Omega^3{}_1 = \Big(\frac{m'}{R'R^2} - \frac{m}{R^3}\Big)\,\theta^1 \wedge \theta^3 && \Omega^2{}_3 = -\Omega^3{}_2 = 2mR^{-3}\,\theta^2 \wedge \theta^3.
\end{aligned} \tag{133}$$

Integrating (124), we get the following implicit expression for R.

$$\Big(\frac{R}{r}\Big)^{3/2} F(pR/r) = F(p) - \frac{t}{t_0}\Big(\frac{a}{a_0}\Big)^{1/2} F(p_0) \tag{134}$$

where $a_0 := a(0) = \rho(0) > 0$, $p_0 := p(0) \leq 1$, $t_0 := (3/8\pi a_0)^{1/2} F(p_0)$, and $F : (-\infty, 1) \to (0, \pi/2)$ is a positive, bounded, smooth, strictly increasing and strictly convex function. If $E(r) < 0$, (134) is singular on a hypersurface $\{t = t_b(r)\}$ where $pR = r$, with $t_b(r) \leq 0$. For $t < t_b(r)$ an equation similar to (134) holds, and we will concentrate on the region where $t > 0$. We refer to [11] for the details.

There are several types of singularities in the Tolman-Bondi space-time. There is a coordinate singularity at $r = 0$, a central singularity at $(t, r) = (t_0, 0)$, and a final singularity at $r > 0$, $R = 0$. For some parameter values, there are also shell crossing singularities where $R' = 0$ (see §7.2 below).

First we study the final singularity. Let γ be a curve with constant r, ϑ and ϕ, and parameterise γ by $\tau := t_s - t$, where

$$t_s := \left(\frac{a}{a_0}\right)^{1/2} \frac{F(p)}{F(p_0)}\, t_0. \tag{135}$$

Then γ starts at the final singularity at $\tau = 0$ and E is parallel along γ. All functions not depending on t are bounded, so from (134) and (124), there are constants N_1, N_2 and N_3 such that

$$\|\mathbf{R}\| < N_1\, \tau^{-2} \tag{136}$$

$$\|\nabla\mathbf{R}\| < N_2\, \tau^{-3} \tag{137}$$

$$\|\mathbf{R}^{-1}\| < N_3\, \tau^{2}. \tag{138}$$

By an argument as in §6.1, fibres over the final singularity are degenerate.

Next we turn our attention to the central singularity at $(t_0, 0)$. Let γ be a radial curve with $t = t_0$ and constant ϑ and ϕ, parameterised by r and starting at $(t, r) = (t_0, 0)$. Also, let $F = E\mathbf{L}$ be parallel along γ. Solving the parallel propagation equation we find that $\mathbf{L}$ is a Lorentz boost in the (E_0, E_1)-plane with hyperbolic angle

$$\varphi := -\int \frac{\dot{R}'}{W}\, \mathrm{d}r. \tag{139}$$

Let

$$C_0 := \frac{1}{2} p''(0) F'(p_0) - \frac{1}{4a_0} a''(0) F(p_0). \tag{140}$$

We assume that $C_0 \neq 0$, the case of interest being $C_0 > 0$ since then the singularity is naked [11]. If we restrict attention to the neighbourhood of γ where

$$|t - t_0| < \frac{C_0\, t_0}{3F(p_0)} r^2, \tag{141}$$

then it is possible to use (134), (124) and the fact that a and p extends to $\mathbb{R}$ to estimate all components of Γ, $\mathbf{R}$ and $\nabla\mathbf{R}$. We find that there are positive constants N_1, N_2 and N_3 such that

$$\|\mathbf{R}\| < N_1\, r^{-4} \tag{142}$$

$$\|\nabla\mathbf{R}\| < N_2\, r^{-19/3} \tag{143}$$

$$\|\mathbf{R}^{-1}\| < N_3\, r^{4}. \tag{144}$$

Also, φ is bounded as $r \to 0$. Again, an argument similar to the one in §6.1, with ρ_i proportional to $r_i^{7/3}$, gives that the fibre is totally degenerate also for this naked singularity. Note that $\|\nabla\mathbf{R}\|$ has a stronger divergence than $\|\mathbf{R}\|^{3/2}$.

7 Partial degeneracy

In general it can be very hard to show that a boundary fibre is degenerate, since different subgroups of the singular holonomy group may be generated by various things, e.g. unbounded curvature, regular curvature, quasi-regular singularities, and contributions from other boundary points due to non-Hausdorff behaviour of the b-boundary [2]. Note that even if the Riemann tensor is non-invertible and/or if only some components diverge, in some cases Lemma 3 may be used to establish partial degeneracy at least.

7.1 Quasi-regular singularities

To illustrate how degeneracy can be caused by topological anomalies we consider quasi-regular singularities obtained by suitable identifications in (the universal covering space of) Minkowski space-time (M, g) with

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{145}$$

Given an isometry φ, we may identify points $\varphi(p)$ with p in (the universal covering space of) a subset of (M, g) [5].

As a first example, let $(\hat{M}, \hat{g})$ be the universal covering space of Minkowski space with the timelike 2-plane $\{x = y = 0\}$ removed and let φ be the rotation in the (x, y)-plane by an angle $\phi \neq 2\pi$. Then the space-time obtained by identifying points with their images under φ has a conelike singularity at $\{x = y = 0\}$. Since $(\hat{M}, \hat{g})$ is flat, the infinitesimal, local and restricted holonomy groups are all trivial, so the only contribution to the singular holonomy group comes from curves not homotopic to 0. It clearly suffices to study curves with $x^2 + y^2 = r$ as $r \to 0$, and a simple argument then gives that Φ^s_{OM} is a discrete group generated by ϕ modulo 2π.

Secondly, let φ be a boost in the (t, x)-plane with hyperbolic angle ϕ and consider the subset $\{z > -t\}$ of (M, g). Identifying points under φ we get the Misner space-time with quasi-regular singularities similar to the ones in the Taub-NUT space-time [6, 9]. As for the conelike example above, it is straightforward to show that Φ^s_{OM} is generated by φ. More complicated singularities can be constructed by variations of this procedure [5].

7.2 Shell crossing singularities

We return to the Tolman-Bondi space-time from §6.5 to study the shell crossing singularities where $R' = 0$. Only some components of the curvature diverge, so all we can hope for is to establish partial degeneracy in some directions. Unfortunately, it turns out that while $\|\mathbf{R}\|$ is of order $(R')^{-1}$, $\|\nabla\mathbf{R}\|$ is of order $(R')^{-3}$, which prohibits us from using Lemma 3 in this case. Also, higher order derivatives of the Riemann tensor have even stronger divergence. Since the infinitesimal holonomy group is generated by the Riemann tensor and its derivatives, whose norms all diverge, it seems probable that the singular holonomy group is nontrivial. Proving that is impossible with our technique however, since we have no way to control the contributions from higher order terms.

8 Discussion

We have shown that in many cases, the b-boundary have totally degenerate fibres, leading to undesired topological effects. The argument is based on that the divergence of the derivative of the Riemann tensor is sufficiently weak, so that the essential contribution to the singular holonomy group comes from $\mathbf{R}(\mathbf{X}, \mathbf{Y})$. As we saw in §7.2, this fails in some cases. Since the infinitesimal holonomy group is generated by expressions of the form $\nabla_{\mathbf{V}_1 \ldots \mathbf{V}_n}(\mathbf{X}, \mathbf{Y})$, it might be possible to use higher order derivatives of the Riemann tensor to generate elements in the singular holonomy group. One would then have to go further in the expansion in the proof of Lemma 2, and the conditions would get much more complicated.

In §7.1, we gave a simple example of how a quasi-regular singularity can give rise to degenerate fibres. It is very easy to construct examples of quasi-regular singularities with discrete singular holonomy groups, but it is unknown if nondiscrete groups can arise in this way.

The most apparent unsolved problem involving the b-boundary is the structure of the boundary itself. In the FLRW case the boundary has been shown to be a single point [3]. But for the Schwarzschild space-time, the results are not as conclusive. Both Bosshard [1] and Johnson [7] have established partial degeneracy of boundary fibres, but it is unknown if the boundary is just a point or something else (Johnson conjectures that it is a line).

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